Hardness Results on Steiner Type Packing Problems in Digraphs

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Outline

Steiner Tree

For a graph $G = (V, E)$ and a set of terminal vertices $S \subseteq V$, an S-Steiner tree (or, S-tree) is a tree T of G with $S \subseteq V(T)$.

Two S-trees are said to be edge-disjoint if they have no common edge. Two edge-disjoint S-trees T_1 and T_2 are said to be internally disjoint if $V(T_1) \cap V(T_2) = S$.

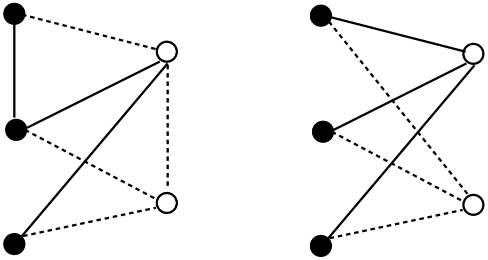

Figure: Two examples.

Steiner Tree Packing problem

Steiner tree packing problem (STP): The input consists of an undirected graph G and a subset of vertices $S \subseteq V(G)$, the goal is to find a largest collection of edge-disjoint S-Steiner trees.

Besides the classical version, people also study some other variations, such as packing internally disjoint Steiner trees, packing directed Steiner trees and packing strong subgraphs.

Applications

VLSI circuit design:

M. Grötschel, A. Martin, R. Weismantel, The Steiner tree packing problem in VLSI design, Math. Program. 78(1997), 265–281.

N. Sherwani, Algorithms for VLSI Physical Design Automation, 3rd Edition, Kluwer Acad. Pub., London, 1999.

Computer communication networks:

D. Du, X. Hu, Steiner Tree Problems in Computer Communication Networks, World Scientific, 2008.

Optical wireless communication networks:

X. Cheng, D. Du, Steiner Trees in Industry, Kluwer Academic Publisher, Dordrecht, 2001.

Publications on Steiner tree packing

1. W.R. Pulleyblank, Two Steiner tree packing problems, STOC, pp. 383–387, 1995.

2. M. Grötschel, A. Martin, R. Weismantel, Packing Steiner trees: further facets, European J. Combin. 17, 1996, 39–52.

3. M. Grötschel, A. Martin, R. Weismantel, Packing Steiner trees: polyhedral investigations, Mathematical Programming 72, 1996, 101–123.

4. M. Grötschel, A. Martin, R. Weismantel, Packing Steiner trees: a cutting plane algorithm and computational results, Mathematical Programming 72, 1996, 125–145.

5. M. Grötschel, A. Martin, R. Weismantel, Packing Steiner trees: separation algorithms, SIAM J. Discrete Math. 9, 1996, 233–257.

6. M. Grötschel, A. Martin, R. Weismantel, The Steiner tree packing problem in VLSI design, Math. Program. 78, 1997, 265–281.

Publications on Steiner tree packing

7. E. Uchoa and M. Poggi de Aragão, Vertex-disjoint packing of two Steiner trees: polyhedra and branch-and-cut, Mathematical Programming 90, 2001, 537–557.

8. G.W. Jeong, K. Lee, S. Park and K. Park. A branch-and-price algorithm for the Steiner tree packing problem, Computers and Operations Research 29, 2002, 221–241.

9. A. Frank, T. Király and M. Kriesell, On decomposing a hypergraph into k connected sub-hypergraphs, Discrete Applied Mathematics 131, 2003, 373–383.

10. K. Jain, M. Mahdian and M.R. Salavatipour: Packing Steiner trees, SODA, pp. 266–274, 2003.

11. M. Kriesell, Edge-disjoint trees containing some given vertices in a graph, J. Combin. Theory Ser. B 88, 2003, 53–65.

12. J. Cheriyan and M. Salavatipour, Hardness and approximation results for packing Steiner trees, Algorithmica, 45, 2006, 21–43.

Publications on Steiner tree packing

13. L. Lau, An approximate max-Steiner-tree-packing min-Steiner-cut theorem, Combinatorica 27, 2007, 71–90.

14. M. Kriesell, Edge disjoint Steiner trees in graphs without large bridges, J. Graph Theory 62, 2009, 188–198.

15. D. West, H. Wu, Packing Steiner trees and S-connectors in graphs, J. Combin. Theory Ser. B 102, 2012, 186–205.

16. M. DeVos, J. McDonald, I. Pivotto, Packing Steiner trees, J. Combin. Theory Ser. B, 119, 2016, 178–213.

17. X. Li and Y. Mao, Generalized Connectivity of Graphs, Springer, Switzerland, 2016.

Directed Steiner Tree

An out-tree is an oriented tree in which every vertex except one, called the root, has in-degree one. For a digraph $D = (V(D), A(D))$, and a set $S \subseteq V(D)$ with $r \in S$ and $|S| \geq 2$, a directed (S, r)-Steiner tree or, simply, an (S, r)-tree is an out-tree T rooted at r with $S \subseteq V(T)$.

Two (S, r)-trees T_1 and T_2 are said to be arc-disjoint if $A(T_1) \cap A(T_2) = \emptyset$. Two arc-disjoint (S, r)-trees T_1 and T_2 are said to be internally disjoint if $V(T_1) \cap V(T_2) = S$.

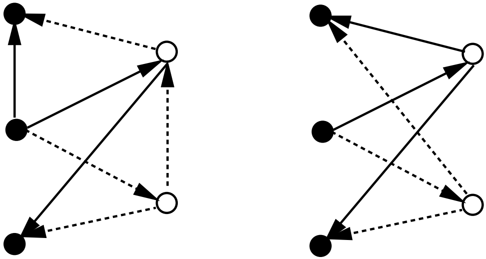

Figure: Two examples.

Directed Steiner Tree Packing problem

Cheriyan and Salavatipour (Algorithmica, 2006) introduced and studied the following two directed Steiner tree packing problems.

Arc-disjoint directed Steiner tree packing problem (ADSTP): The input consists of a digraph D and a subset of vertices $S \subseteq V(D)$ with a root r, the goal is to find a largest collection of arc-disjoint (S, r)-trees.

Internally-disjoint directed Steiner tree packing problem (IDSTP): The input consists of a digraph D and a subset of vertices $S \subseteq V(D)$ with a root r, the goal is to find a largest collection of internally disjoint (S, r)-trees.

Local parameters: $\kappa_{S,r}(D)$ *and* $\lambda_{S,r}(D)$

Let $\kappa_{S,r}(D)$ (respectively, $\lambda_{S,r}(D)$) be the maximum number of internally disjoint (respectively, arc-disjoint) (S, r)-trees in D.

$|S| = 2$: $\kappa_{S,r}(D)$ (respectively, $\lambda_{S,r}(D)$) is the maximum number of internally disjoint (respectively, arc-disjoint) $r - x$ paths in D, where $S = \{r, x\}$ (Menger's Theorem).

$|S| = n$: $\kappa_{S,r}(D) = \lambda_{S,r}(D)$ is the maximum number of arc-disjoint out-branchings rooted at r in D (Edmonds' Branching Theorem).

Global parameters: $\kappa_k(D)$ and $\lambda_k(D)$

Sun and Yeo (arXiv:2005.00849v3) introduced the following concept of directed tree connectivity which is related to directed Steiner tree packing problem and is a natural extension of tree connectivity of undirected graphs to directed graphs.

The generalized k-vertex-strong connectivity of D is defined as

$$\kappa_k(D) = \min\{\kappa_{S,r}(D) \mid S \subset V(D), |S| = k, r \in S\}.$$

Similarly, the generalized k-arc-strong connectivity of D is defined as

$$\lambda_k(D) = \min\{\lambda_{S,r}(D) \mid S \subset V(D), |S| = k, r \in S\}.$$

By definition, when $k = 2$, $\kappa_2(D) = \kappa(D)$ and $\lambda_2(D) = \lambda(D)$. Hence, these two parameters could be seen as generalizations of vertex-strong connectivity and arc-strong connectivity of a digraph. The generalized k-vertex-strong connectivity and k-arc-strong connectivity are also called directed tree connectivity.

Papers on directed Steiner tree packing

Cheriyan and Salavatipour, Hardness and approximation results for packing Steiner trees, Algorithmica, 45, 2006, 21–43.

Sun and Yeo, Directed Steiner tree packing and directed tree connectivity, arXiv:2005.00849v3 [math.CO].

S-strong subgraphs

Note that an S-tree is a connected subgraph of G containing S. In fact, in the problem of Steiner tree packing, we could replace "edge-disjoint S-trees" by "edge-disjoint connected subgraphs of G containing S (or, simply, S-connected subgraph)". It is natural to extend the problem of Steiner tree packing to digraphs.

Let $D = (V(D), A(D))$ be a digraph of order n, $S \subseteq V$ a k-subset of $V(D)$ and $2 \leq k \leq n$. A subdigraph H of D is called an S-strong subgraph if H is strong and $S \subseteq V(H)$.

Two S-strong subgraphs D_1 and D_2 are said to be arc-disjoint if $A(D_1) \cap A(D_2) = \emptyset$. Two arc-disjoint S-strong subgraphs D_1 and D_2 are said to be internally disjoint if $V(D_1) \cap V(D_2) = S$.

Strong subgraph packing problem

Arc-disjoint strong subgraph packing problem(ASSP): The input consists of a digraph D and a subset of vertices $S \subseteq V(D)$, the goal is to find a largest collection of arc-disjoint S-strong subgraphs.

Internally disjoint strong subgraph packing problem(ISSP): The input consists of a digraph D and a subset of vertices $S \subseteq V(D)$, the goal is to find a largest collection of internally disjoint S-strong subgraphs.

Parameters: $\kappa_S(D)$, $\lambda_S(D)$, $\kappa_k(D)$ and $\lambda_k(D)$

Local parameters: let $\kappa_S(D)$ (respectively, $\lambda_S(D)$) be the maximum number of internally disjoint (respectively, arc-disjoint) S-strong subgraphs in D.

Global parameters: The strong subgraph k-connectivity is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V(D), |S| = k\}.$$

Similarly, the strong subgraph k-arc-connectivity is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$

The strong subgraph connectivity is not only a natural extension of tree connectivity of undirected graphs to directed graphs, but also could be seen as a generalization of classical connectivity of undirected graphs by the fact that $\kappa_2(\overleftrightarrow{G}) = \kappa(G)$ and $\lambda_2(\overleftrightarrow{G}) = \lambda(G)$.

Papers on strong subgraph packing problem

Sun, Gutin, Yeo and Zhang, Strong subgraph k-connectivity, J. Graph Theory 92(1), 2019, 5–18.

Sun and Gutin, Strong subgraph connectivity of digraphs, Graphs Combin. 37, 2021, 951–970.

Sun, Gutin and Zhang, Packing strong subgraph in digraphs, arXiv:2110.12783 [math.CO].

Edmonds' Branching Theorem

Edmonds (Edge-disjoint branchings, in *Combinatorial Algorithms* (B. Rustin ed.), 1973.) characterized digraphs having k arc-disjoint out-branchings rooted at a specified root s:

A digraph $D = (V, A)$ with a special vertex s has k arc-disjoint out-branchings rooted at s if and only if there are k arc-disjoint (s, v)-paths in D for every $v \in V - s$.

Furthermore, there exists a polynomial algorithm for finding k arc-disjoint out-branchings from a given root s if they exist. However, it is NP-complete to decide whether there is a pair of arc-disjoint in- and out-branchings B_s^+, B_s^- rooted at s.

A conjecture by Thomassen

In connection with the above problem, Thomassen (Ann. New York Academy Sci., 1989) posed the following conjecture:

Conjecture

There exists an integer N so that every N-arc-strong digraph D contains a pair of arc-disjoint in- and out-branchings.

Strong arc decomposition

A digraph $D = (V, A)$ has a strong arc decomposition if A has two disjoint sets A_1 and A_2 such that both (V, A_1) and (V, A_2) are strong. Bang-Jensen and Yeo (Combinatorica, 2004) posed a stronger conjecture:

Conjecture

There exists an integer N so that every N-arc-strong digraph D has a strong arc decomposition.

They proved that it is NP-complete to decide whether a digraph has a strong arc decomposition.

Papers on strong arc decomposition

Observe that a digraph D has a strong arc decomposition if and only if $\lambda_{V(D)}(D) \geq 2$ (or, $\kappa_{V(D)}(D) \geq 2$).

Bang-Jensen and Yeo, Decomposing k-arc-strong tournaments into strong spanning subdigraphs, Combinatorica 24(3), 2004, 331–349.

Bang-Jensen and Huang, Decomposing locally semicomplete digraphs into strong spanning subdigraphs, J. Combin. Theory Ser. B, 102, 2012, 701–714.

Sun, Gutin and Ai, Arc-disjoint strong spanning subdigraphs in compositions and products of digraphs, Discrete Math. 342(8), 2019, 2297–2305.

Bang-Jensen, Gutin and Yeo, Arc-disjoint strong spanning subgraphs of semicomplete compositions, J. Graph Theory 95(2), 2020, 267–289.

Outline

Directed k-Linkage problem

Directed k-Linkage problem: Given a digraph D and a (terminal) sequence $(s_1, t_1, \ldots, s_k, t_k)$ of distinct vertices of D, decide whether D has k vertex-disjoint paths $P_1, \ldots, P_k$, where P_i starts at s_i and ends at t_i for all $i \in [k]$.

Complexity for Directed k-Linkage problem

*NP-complete:

When $k = 2$ for general digraphs (Fortune, Hopcroft and Wyllie, Theoret. Comput. Sci., 1980),
even restricted to Eulerian digraphs (Sun and Yeo, arXiv:2005.00849v3);
Tournaments when k is not fixed (Bang-Jensen and Thomassen, SIAM J. Discrete Math., 1992).

Complexity for Directed k-Linkage problem

*Polynomial (for fixed k):

Acyclic digraphs (Fortune, Hopcroft and Wyllie, Theoret. Comput. Sci., 1980);

Digraphs of bounded directed tree-width (Johnson, Robertson, Seymour and Thomas, J. Combin. Theory Ser. B, 2001);

Semicomplete digraphs (Chudnovsky, Scott and Seymour, Adv. Math., 2015);

Locally semicomplete digraphs and several classes of decomposable digraphs (Bang-Jensen, Christiansen and Maddaloni, J. Graph Theory, 2017);

Digraphs which have a p-partition (p is fixed) each part of which is semicomplete (Chudnovsky, Scott and Seymour, J. Combin. Theory Ser. B, 2019).

Directed Weak k-Linkage problem

The DIRECTED WEAK k-LINKAGE PROBLEM is formulated as follows: for a fixed integer $k \geq 2$, given a digraph D and a (terminal) sequence $((s_1, t_1), \dots, (s_k, t_k))$ of distinct vertices of D, decide whether D has k arc-disjoint paths $P_1, \dots, P_k$, where P_i starts at s_i and ends at t_i for all $i \in [k]$.

Complexity for Directed Weak k-Linkage problem

*NP-complete:

When $k = 2$ for general digraphs (Fortune, Hopcroft and Wyllie, Theoret. Comput. Sci., 1980);

Eulerian digraphs when k is not fixed (Ibaraki and Poljak, SIAM J. Discrete Math., 1991).

Complexity for Directed Weak k-Linkage problem

*Polynomial (for fixed k):

Acyclic digraphs (Fortune, Hopcroft and Wyllie, Theoret. Comput. Sci., 1980);
Semicomplete digraphs (Bang-Jensen, J. Combin. Theory Ser. B, 1991);
Digraphs with bounded independence number (Fradkin and Seymour, J. Combin. Theory Ser. B, 2015).

Outline

General digraphs

Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$. It is natural to consider the following problem:

What is the complexity of deciding whether $\kappa_{S,r}(D) \geq \ell$ (resp. $\lambda_{S,r}(D) \geq \ell$)? where $r \in S$ is a root.

If $k = 2$, say $S = \{r, x\}$, then the problem of deciding whether $\kappa_{S,r}(D) \geq \ell$ (resp. $\lambda_{S,r}(D) \geq \ell$) is equivalent to deciding whether $\kappa(r, x) \geq \ell$ (resp. $\lambda(r, x) \geq \ell$), and so is polynomial-time solvable.

If $\ell = 1$, then the above problem is also polynomial-time solvable by the well-known fact that every strong digraph has an out- and in-branching rooted at any vertex, and these branchings can be found in polynomial-time.

General digraphs

Hence, it remains to consider the case that $k \geq 3, \ell \geq 2$. Using the NP-completeness of DIRECTED 2-LINKAGE PROBLEM for general digraphs, Cheriyan and Salavatipour (Algorithmica, 2006) proved the NP-hardness of the case $k = 3, \ell = 2$ for both $\kappa_{S,r}(D)$ and $\lambda_{S,r}(D)$:

(a): Let D be a digraph and $S \subseteq V(D)$ with $|S| = 3$. The problem of deciding whether $\kappa_{S,r}(D) \geq 2$ is NP-hard, where $r \in S$.

(b): Let D be a digraph and $S \subseteq V(D)$ with $|S| = 3$. The problem of deciding whether $\lambda_{S,r}(D) \geq 2$ is NP-hard, where $r \in S$.

General digraphs

Sun and Yeo (arXiv:2005.00849v3) extended the above results to the following:

Let $k \geq 3$ and $\ell \geq 2$ be fixed integers (considered as constants). Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$ and $r \in S$. Both the following problems are NP-complete.

- Is $\kappa_{S,r}(D) \geq \ell$?
- Is $\lambda_{S,r}(D) \geq \ell$?

Complexity for $\lambda_{S,r}(D)$ on general digraphs

Table 1: Directed graphs

$\lambda_{S,r}(D) \geq \ell$? $\lvert S\rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	NP-complete	NP-complete	NP-complete
$\ell \geq 3$ constant	NP-complete	NP-complete	NP-complete
ℓ part of input	NP-complete	NP-complete	NP-complete

Cheriyan and Salavatipour, Hardness and approximation results for packing Steiner trees, Algorithmica, 45, 2006, 21–43.

Sun and Yeo, Directed Steiner tree packing and directed tree connectivity, arXiv:2005.00849v3 [math.CO].

Complexity for $\kappa_{S,r}(D)$ on general digraphs

Table 2: Directed graphs

$\kappa_{S,r}(D) \geq \ell$? $\lvert S\rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	NP-complete	NP-complete	NP-complete
$\ell \geq 3$ constant	NP-complete	NP-complete	NP-complete
ℓ part of input	NP-complete	NP-complete	NP-complete

Cheriyan and Salavatipour, Hardness and approximation results for packing Steiner trees, Algorithmica, 45, 2006, 21–43.

Sun and Yeo, Directed Steiner tree packing and directed tree connectivity, arXiv:2005.00849v3 [math.CO].

Bang-Jensen, Frank and Jackson (SIAM J. Discrete Math., 1995) proved the following result:

Let $k \geq 1$ and let $D = (V, A)$ be a directed multigraph with a special vertex z. Let $T' = \{x \mid x \in V \setminus \{z\} \text{ and } d^-(x) < d^+(x)\}$. If $\lambda(z, x) \geq k$ for every $x \in T'$, then there exists a family $\mathcal{F}$ of k arc-disjoint out-trees rooted at z so that every vertex $x \in V$ belongs to at least $\min\{k, \lambda(z, x)\}$ members of $\mathcal{F}$.

In the case when D is Eulerian, then $d^+(x) = d^-(x)$ for all $x \in V(D)$ and therefore $T' = \emptyset$ in the above theorem. Therefore the following corollary holds:

Let $k \geq 1$ and let $D = (V, A)$ be an Eulerian digraph with a special vertex z. Then there exists a family $\mathcal{F}$ of k arc-disjoint out-trees rooted at z so that every vertex $x \in V$ belongs to at least $\min\{k, \lambda(z, x)\}$ members of $\mathcal{F}$.

Eulerian digraphs

Using the above result, Sun and Yeo (arXiv:2005.00849v3) proved:

If D is an Eulerian digraph and $S \subseteq V(D)$ and $r \in S$, then $\lambda_{S,r}(D) \geq \ell$ if and only if $\lambda_D(r, s) \geq \ell$ for all $s \in S \setminus \{r\}$.

Eulerian digraphs

Using the NP-completeness of Directed 2-Linkage problem restricted to Eulerian digraphs, Sun and Yeo (arXiv:2005.00849v3) further proved:

Let $k \geq 3$ and $\ell \geq 2$ be fixed integers. Let D be an Eulerian digraph and $S \subseteq V(D)$ with $|S| = k$ and $r \in S$. Then deciding whether $\kappa_{S,r}(D) \geq \ell$ is NP-complete.

Complexity for $\lambda_{S,r}(D)$ on Eulerian digraphs

Table 3: Eulerian digraphs

$\lambda_{S,r}(D) \geq \ell$? $\lvert S\rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	Polynomial	Polynomial	Polynomial
$\ell \geq 3$ constant	Polynomial	Polynomial	Polynomial
ℓ part of input	Polynomial	Polynomial	Polynomial

Sun and Yeo, Directed Steiner tree packing and directed tree connectivity, arXiv:2005.00849v3 [math.CO].

Complexity for $\kappa_{S,r}(D)$ on Eulerian digraphs

Table 4: Eulerian digraphs			
$\kappa_{S,r}(D) \geq \ell$? $\lvert S \rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	NP-complete	NP-complete	NP-complete
$\ell \geq 3$ constant	NP-complete	NP-complete	NP-complete
ℓ part of input	NP-complete	NP-complete	NP-complete

Sun and Yeo, Directed Steiner tree packing and directed tree connectivity, arXiv:2005.00849v3 [math.CO].

Robertson and Seymour (J. Combin. Theory Ser. B, 1995) proved the following result on the k-Linkage problem of general graphs: Let G be a graph and let $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ be $2k$ disjoint vertices in G. We can in $O(|V(G)|^3)$ time decide if there exists an (s_i, t_i)-path, P_i, such that all $P_1, P_2, \ldots, P_k$ are vertex disjoint.

Based on the above result, Sun and Yeo (arXiv:2005.00849v3) proved: Let D be a symmetric digraph and let $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ be vertices in D (not necessarily disjoint) and let $S \subseteq V(D)$. We can in $O(|V(G)|^3)$ time decide if there for all $i = 1, 2, \ldots, k$ exists an (s_i, t_i)-path, P_i, such that no internal vertex of any P_i belongs to S or to any path P_j with $j \neq i$ (the end-points of P_j can also not be internal vertices of P_i).

Symmetric digraphs: both k and ℓ are fixed

Furthermore, Sun and Yeo (arXiv:2005.00849v3) proved:

Let $k \geq 3$ and $\ell \geq 2$ be fixed integers and let D be a symmetric digraph. Let $S \subseteq V(D)$ with $|S| = k$ and let r be an arbitrary vertex in S. Let $A_0, A_1, A_2, \ldots, A_\ell$ be a partition of the arcs in $D[S]$.

We can in time $O(n^{\ell k - 2\ell + 3} \cdot (2k-3)^{\ell(2k-3)})$ decide if there exist ℓ internally disjoint (S, r)-trees, $T_1, T_2, \ldots, T_\ell$, with $A(T_i) \cap A[S] = A_i$ for all $i = 1, 2, \ldots, \ell$ (note that A_0 are the arcs in $D[S]$ not used in any of the trees).

By the above result, they proved:

Let $k \geq 3$ and $\ell \geq 2$ be fixed integers. We can in polynomial time decide if $\kappa_{S,r}(D) \geq \ell$ for any symmetric digraph, D, with $S \subseteq V(D)$, with $|S| = k$ and $r \in S$.

Symmetric digraphs: k is fixed and ℓ is part of input

Chen, Li, Liu and Mao (J. Combin. Optim., 2017) introduced the following problem, which turned out to be NP-complete:

CLLM PROBLEM: Given a tripartite graph $G = (V, E)$ with a 3-partition (A, B, C) such that $|A| = |B| = |C| = q$, decide whether there is a partition of V into q disjoint 3-sets $V_1, \dots, V_q$ such that for every $V_i = \{a_{i_1}, b_{i_2}, c_{i_3}\}$ $a_{i_1} \in A, b_{i_2} \in B, c_{i_3} \in C$ and $G[V_i]$ is connected.

By the NP-completeness of the CLLM PROBLEM, Sun and Yeo (arXiv:2005.00849v3) proved:

Let $k \geq 3$ be a fixed integer. The problem of deciding if a symmetric digraph D, with a k-subset S of $V(D)$ with $r \in S$ satisfies $\kappa_{S,r}(D) \geq \ell$ (ℓ is part of the input), is NP-complete.

Symmetric digraphs: ℓ is fixed and k is part of input

The 2-Coloring Hypergraphs Problem is defined as the following: Given a hypergraph H with vertex set $V(H)$ and edge set $E(H)$, determine if we can 2-colour the vertices $V(H)$ such that every hyperedge in $E(H)$ contains vertices of both colours. This problem is known to be NP-hard (Lovász, Utilitas Math., 1973).

By constructing a reduction from this problem, Sun and Yeo (arXiv:2005.00849v3) proved:

Let $\ell \geq 2$ be a fixed integer. The problem of deciding if a symmetric digraph D, with a $S \subseteq V(D)$ and $r \in S$ satisfies $\kappa_{S,r}(D) \geq \ell$ ($k = |S|$ is part of the input), is NP-complete.

Complexity for $\lambda_{S,r}(D)$ on symmetric digraphs

Table 5: Symmetric digraphs

$\lambda_{S,r}(D) \geq \ell$? $\lvert S\rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	Polynomial	Polynomial	Polynomial
$\ell \geq 3$ constant	Polynomial	Polynomial	Polynomial
ℓ part of input	Polynomial	Polynomial	Polynomial

Sun and Yeo, Directed Steiner tree packing and directed tree connectivity, arXiv:2005.00849v3 [math.CO].

Complexity for $\kappa_{S,r}(D)$ on symmetric digraphs

Table 6: Symmetric digraphs

$\kappa_{S,r}(D) \geq \ell$? $\lvert S\rvert = k$	$k = 3$	$k \geq 4$ constant	k part of input
$\ell = 2$	Polynomial	Polynomial	NP-complete
$\ell \geq 3$ constant	Polynomial	Polynomial	NP-complete
ℓ part of input	NP-complete	NP-complete	NP-complete

Sun and Yeo, Directed Steiner tree packing and directed tree connectivity, arXiv:2005.00849v3 [math.CO].

Hardness of approximation of ADSTP *and* IDSTP

For general digraphs, Cheriyan and Salavatipour (Algorithmica, 2006) studied the hardness of approximation of both ADSTP and IDSTP as follows:

Given an instance of ADSTP, it is NP-hard to approximate the solution within $O(m^{1/3-\epsilon})$ for any $\epsilon > 0$.

Given an instance of IDSTP, it is NP-hard to approximate the solution within $O(n^{1/3-\epsilon})$ for any $\epsilon > 0$.

Two more general problems: GDE and GDV

GDE: The input consists of a digraph D, a capacity c_e on each arc $e \in A(D)$, ℓ terminal sets $T_1, \dots, T_\ell$ and ℓ roots $r_1, \dots, r_\ell$ such that $r_i \in V(T_i)$ for each $i \in [\ell]$. The goal is to find a largest collection of directed Steiner trees, each rooted at an r_i and containing all vertices of T_i such that each arc e is contained in at most c_e directed trees.

GDV: The input consists of a digraph D, a capacity c_v on each vertex $v \in V(D)$, ℓ terminal sets $T_1, \dots, T_\ell$ and ℓ roots $r_1, \dots, r_\ell$ such that $r_i \in V(T_i)$ for each $i \in [\ell]$. The goal is to find a largest collection of directed Steiner trees, each rooted at an r_i and containing all vertices of T_i such that each non-terminal vertex v is contained in at most c_v directed trees.

Two more general problems: GDE *and* GDV

When $\ell = 1$ and $c_e = 1$ (resp. $c_v = 1$), then GDE (resp. GDV) is exactly ADSTP (resp. IDSTP). The hardness of approximation of both GDE and GDV has also been studied in

V. Guruswami, S. Khanna, R. Rajaraman, B. Shepherd, and M. Yannakakis, Near-optimal hardness results and approximation algorithms for edge-disjoint paths and related problems, J. Comput. System Sci. 67(3), 2003, 473–496.

J. Cheriyan and M. Salavatipour, Hardness and approximation results for packing Steiner trees, Algorithmica, 45, 2006, 21–43.

Outline

Complexity for $\kappa_S(D)$ on general digraphs

By using the reduction from the DIRECTED 2-LINKAGE PROBLEM on general digraphs, Sun, Gutin, Yeo and Zhang (J. Graph Theory, 2019) proved the following intractability result: Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$. The problem of deciding whether $\kappa_S(D) \geq \ell$ is NP-complete.

Table A: Directed graphs		
$\lambda_S(D) \geq \ell$? $\lvert S\rvert = k$	$k \geq 2$ constant	k part of input
$\ell \geq 2$ constant	NP-complete	NP-complete
ℓ part of input	NP-complete	NP-complete

Complexity for $\lambda_S(D)$ on general digraphs

By using the reduction from the Directed Weak 2-Linkage problem of general digraphs, Sun and Gutin (Graphs Combin., 2021) proved: Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$. The problem of deciding whether $\lambda_S(D) \geq \ell$ is NP-complete.

Table B: Directed graphs		
$\kappa_S(D) \geq \ell$? $\lvert S\rvert = k$	$k \geq 2$ constant	k part of input
$\ell \geq 2$ constant	NP-complete	NP-complete
ℓ part of input	NP-complete	NP-complete

Semicomplete digraphs

Chudnovsky, Scott and Seymour (J. Combin. Theory Ser. B, 2019) proved the following powerful result: Let k and c be fixed positive integers. Then the DIRECTED k-LINKAGE PROBLEM on a digraph D whose vertex set can be partitioned into c sets each inducing a semicomplete digraph and a terminal sequence $((s_1, t_1), \dots, (s_k, t_k))$ of distinct vertices of D, can be solved in polynomial time.

By the above result, Sun, Gutin, Yeo and Zhang (J. Graph Theory, 2019) proved: Let k and ℓ be fixed positive integers. Let D be a digraph and let $X_1, X_2, \dots, X_\ell$ be ℓ vertex disjoint subsets of $V(D)$, such that $|X_i| \leq k$ for all $i \in [\ell]$. Let $X = \cup_{i=1}^{\ell} X_i$ and assume that every vertex in $V(D) \setminus X$ is adjacent to every other vertex in D. Then we can in polynomial time decide if there exists vertex disjoint subsets $Z_1, Z_2, \dots, Z_\ell$ of $V(D)$, such that $X_i \subseteq Z_i$ and $D[Z_i]$ is strongly connected for each $i \in [\ell]$.

Complexity for $\kappa_S(D)$ on semicomplete digraphs

Furthermore, Sun, Gutin, Yeo and Zhang (J. Graph Theory, 2019) proved: Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$. The problem of deciding whether $\kappa_S(D) \geq \ell$ for a semicomplete digraph D is polynomial-time solvable.

Table C: Semicomplete digraphs			
$\kappa_S(D) \geq \ell$? $\|S\| = k$	$k = 2$	$k \geq 3$ constant	k part of input
$\ell \geq 2$ constant	Polynomial	Polynomial	
ℓ part of input			

Questions

The DIRECTED k-LINKAGE PROBLEM is polynomial-time solvable for planar digraphs (Schrijver, SIAM J. Comput., 1994) and digraphs of bounded directed treewidth (Johnson, Robertson, Seymour and Thomas, J. Combin. Theory Ser. B, 2001). However, it seems that we cannot use the similar approach directly as the structure of minimum-size strong subgraphs in these two classes of digraphs is more complicated than in semicomplete digraphs. Sun, Gutin, Yeo and Zhang (J. Graph Theory, 2019) posed the following questions:

Problem

What is the complexity of deciding whether $\kappa_S(D) \geq \ell$ for (fixed) integers $k \geq 2$ and $\ell \geq 2$, and a planar digraph D (respectively, a digraph D of bounded directed treewidth), where $S \subset V(D)$ with $|S| = k$?

Symmetric digraphs: k is fixed but ℓ is a part of input

Chen, Li, Liu and Mao (J. Combin. Optim., 2017) introduced the following problem, which turned out to be NP-complete:

CLLM Problem: Given a tripartite graph $G = (V, E)$ with a 3-partition (A, B, C) such that $|A| = |B| = |C| = q$, decide whether there is a partition of V into q disjoint 3-sets $V_1, \dots, V_q$ such that for every $V_i = \{a_{i_1}, b_{i_2}, c_{i_3}\}$ $a_{i_1} \in A, b_{i_2} \in B, c_{i_3} \in C$ and $G[V_i]$ is connected.

By this result, Sun, Gutin, Yeo and Zhang (J. Graph Theory, 2019) proved: For any fixed integer $k \geq 3$, given a symmetric digraph D, a k-subset S of $V(D)$ and an integer ℓ ($\ell \geq 1$), deciding whether $\kappa_S(D) \geq \ell$, is NP-complete.

Symmetric digraphs: k and ℓ are fixed

Sun, Gutin, Yeo and Zhang (J. Graph Theory, 2019) proved the following technical lemma: Let $k, \ell \geq 2$ be fixed. Let G be a graph and let $S \subseteq V(G)$ be an independent set in G with $|S| = k$. For $i \in [\ell]$, let D_i be any set of arcs with both end-vertices in S. Let a forest F_i in G be called (S, D_i)-*acceptable* if the digraph $\overleftrightarrow{F_i} + D_i$ is strong and contains S. In polynomial time, we can decide whether there exists edge-disjoint forests $F_1, F_2, \ldots, F_\ell$ such that F_i is (S, D_i)-acceptable for all $i \in [\ell]$ and $V(F_i) \cap V(F_j) \subseteq S$ for all $1 \leq i < j \leq \ell$.

They further proved the following result by the above lemma: Let $k, \ell \geq 2$ be fixed. For any symmetric digraph D and $S \subseteq V(D)$ with $|S| = k$ we can in polynomial time decide whether $\kappa_S(D) \geq \ell$.

Symmetric digraphs: ℓ is fixed but k is a part of input

The 2-Coloring Hypergraphs Problem is defined as the following: Given a hypergraph H with vertex set $V(H)$ and edge set $E(H)$, determine if we can 2-colour the vertices $V(H)$ such that every hyperedge in $E(H)$ contains vertices of both colours. This problem is known to be NP-hard (Lovász, Utilitas Math., 1973).

By constructing a reduction from this problem, Sun, Gutin and Zhang (arXiv:2110.12783) proved: For any fixed integer $\ell \geq 2$, given a symmetric digraph D, a k-subset S of $V(D)$ and an integer k ($k \geq 2$), deciding whether $\kappa_S(D) \geq \ell$, is NP-complete.

Complexity for $\kappa_S(D)$ on symmetric digraphs

Table D: Symmetric digraphs			
$\kappa_S(D) \geq \ell$? $\lvert S \rvert = k$	$k = 2$	$k \geq 3$ constant	k part of input
$\ell \geq 2$ constant	Polynomial	Polynomial	NP-complete
ℓ part of input	Polynomial	NP-complete	NP-complete

Sun, Gutin, Yeo and Zhang, Strong subgraph k-connectivity, J. Graph Theory 92(1), 2019, 5–18.

Sun, Gutin and Zhang, Packing strong subgraph in digraphs, arXiv:2110.12783 [math.CO].

Complexity for $\kappa_S(D)$ on Eulerian digraphs

Sun and Yeo (arXiv:2005.00849v3) proved: The DIRECTED 2-LINKAGE PROBLEM restricted to Eulerian digraphs is NP-complete.

By this result, Sun, Gutin and Zhang (arXiv:2110.12783) proved: Let $k, \ell \geq 2$ be fixed. For any Eulerian digraph D and $S \subseteq V(D)$ with $|S| = k$, deciding whether $\kappa_S(D) \geq \ell$ is NP-complete.

Table E: Eulerian digraphs

$\kappa_S(D) \geq \ell$? $\lvert S\rvert = k$	$k = 2$	$k \geq 3$ constant	k part of input
$\ell \geq 2$ constant	NP-complete	NP-complete	NP-complete
ℓ part of input	NP-complete	NP-complete	NP-complete

Questions

Problem

What is the complexity of deciding whether $\lambda_S(D) \geq \ell$ for integers $k \geq 3$ and $\ell \geq 2$, and a semicomplete digraph (symmetric digraph, Eulerian digraph) D?

Set cover packing problem

In the SET COVER PACKING PROBLEM, the input consists of a bipartite graph $G = (C \cup B, E)$, and the goal is to find a largest collection of pairwise disjoint set covers of B, where a *set cover* of B is a subset $S \subseteq C$ such that each vertex of B has a neighbor in C.

Feige, Halldorsson, Kortsarz and Srinivasan (SIAM J. Comput., 2002) proved the following inapproximability result on the SET COVER PACKING problem: Unless P=NP, there is no $o(\log n)$-approximation algorithm for SET COVER PACKING, where n is the order of G.

Inapproximability results of ISSP *and* ASSP

Sun, Gutin and Zhang (arXiv:2110.12783) got two inapproximability results on ISSP and ASSP by reductions from the SET COVER PACKING problem:

(a) Unless P=NP, there is no $o(\log n)$-approximation algorithm for ISSP, even restricted to the case that D is a symmetric digraph and S is independent in D, where n is the order of D.

(b) Unless P=NP, there is no $o(\log n)$-approximation algorithm for ASSP, even restricted to the case that S is independent in D, where n is the order of D.

Thanks for your attention!